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Dirac Particles Emission from An Elliptical Black Hole

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ABSTRACT

According to the general theory of relativiy, a black hole is defined as a region of spacetime with super-strong gravitational effects and there is nothing can escape from it. So in the classical theory of relativity, it is safe to say that black hole is a "dead" thermodynamical object. However, by using quantum mechanics theory, Hawking has shown that a black hole may emit particles. In this paper, calculation of temperature of an elliptical black hole when emitting the Dirac particles was presented. By using the complexpath method, radiation can be described as emission process in the tunneling pictures. According to relationship between probability of outgoing particle with the spectrum of black body radiation for fermion particles, temperature of the elliptical black hole can be obtained and it depend on the azimuthal angle. This result also showed that condition on the surface of elliptical black hole is not in thermal equilibrium.

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1. INTRODUCTION

One of the most phenomenal prediction from the general theory of relativity is the existence of black holes. A black hole is defined as a region of spacetime with super-strong gravitational effects and there is nothing can escape from it. Therefore, if studying black holes use only the general relativity, it will lead to an image that black holes are "dead" thermodynamical objects, i.e. they do not have the temperature or entropy. However, in the 1970s, by using quantum mechanics theory, Hawking has shown that black holes are not “so black”. Black holes can radiate particles like in the black body radiation (Hawking, 1974; Hawking, 1975).

The calculation of black hole temperature is a topic that has been widely studied in a recent decade (Tiandho, 2016; Ding & Liu, 2011; Triyanta & Bowaire, 2013; Kai & Shu-Zheng, 2009). However, mostly calculations just focused on the spherical symmetric black holes. The example of discussion about radiation from a non-spherical black hole is radiation from a rotating black hole (Kerr-Newman black hole) (Ma, 2008; Li & Chen, 2015). However, Umetsu’s work show that calculation of temperature in the Kerr-Newman black hole can be simplified just in two dimensional problems (Umetsu, 2010).

In this paper, the radiation of an elliptical black hole was discussed. The purpose of this study was to calculate the temperature of an elliptical black hole if it emit the Dirac particle. The metric used in this study is refered to metric that proposed by Nikouravan (*i.e.* non-rotating elliptical black hole) and it almost similar with the Reissner-Nordstrom black hole (Nikouravan, *et al.*, 2013). To determine its temperature, the complexpath method was used and the particles emission that was analyzed is the Dirac particles. This method is one of the semiclassical methods. This method works according to tunneling mechanism (Siahaan & Triyanta, 2010). In this method, the wavefunction is determined by the ansatz according to its action, and the solution can be obtained through the Hamilton-Jacobi method. By recalling the relation between probability of outgoing particle with spectrum of black body radiation, which radiate fermion, the Hawking temperature can be calculated.

2. METHOD

In this paper, temperature of elliptical black hole if it emits the Dirac particles was calculated using the complex path method. All calculations were made using analytical mathematics.

3. RESULTS AND DISCUSSION

In this section the results and discussion section is divided into several subsections, including review of an elliptical black hole and a process of the Dirac particle emission.

3.1. Review of an elliptical black hole

To get expression of an elliptical metric, Nikouravan uses the general form of static elliptical line element in flat Minkowski spacetime in spherical coordinate (Nikouravan & Rawal, 2011). The expression can be written as

$$ds^2 = e^{2\beta(t,r)}dt^2 - e^{2\gamma(t,r)}\left(\frac{r^2+a^2\cos^2\theta}{r^2+a^2}\right)dr^2 - \left(r^2+a^2\cos^2\theta\right)d\theta^2 - \left(r^2+a^2\right)\sin^2\theta d\varphi^2 \quad (1)$$

where $\beta(t,r)$ and $\gamma(t,r)$ are the unknown function and the expression is determined later and a is a semi-principla axis along x and y axis.

The non-zero components of electromagnetic field tensor is given by,

$$F_{tr} = -F_{rt} = E(r,\theta) \quad (2)$$

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where $E(r,\theta)$ is the electric field as function *r* and θ,

$$E(r,\theta)=\frac{C}{(r^2+a^2)\sin^2\theta+r^2\cos^2\theta} \quad (3)$$

where *C* is a constant and it depend on the boundary condition. According to calculation of energy-momentum tensor and the Ricci tensor, the unkonwn function in **equation** Error! Reference source not found. satisfy condition,

$$\beta(t,r)=-\gamma(r)=\beta(r) \quad (4)$$

Thus, assuming a = 0, the metric in **equation (1)** should reduce to a spherically symmetric metric. In this case the Reissner-Nordstrom metric, it leads to expression as (Nikouravan *et al.*, 2013)

$$\begin{aligned} ds^2 &= \left(1-\frac{2M}{r}+\frac{Q^2}{r^2}\right)dt^2 \\ &-\left(1-\frac{2M}{r}+\frac{Q^2}{r^2}\right)^{-1}\left(\frac{r^2+a^2\cos^2\theta}{r^2+a^2}\right)dr^2 \\ &-\left(r^2+a^2\cos^2\theta\right)d\theta^2 \\ &-\left(r^2+a^2\right)\sin^2\theta d\varphi^2 \end{aligned} \quad (5)$$

From the above metric, the horizon of elliptical black hole that satisfy $g^{rr}=0$ and it can be defined as

$$r_\pm = M\pm\sqrt{M^2-Q^2} \quad (6)$$

In this paper, the horizon that located at $r_\pm=\pm ia$ was ignored since the definition is still disconcerting. Then,

$$P(r)=1-\frac{2M}{r}+\frac{Q^2}{r^2} \quad (7)$$

$$J(r,\theta)=r^2+a^2\cos^2\theta \quad (8)$$

$$K(r)=r^2+a^2 \quad (9)$$

The above metric can be written as

$$\begin{aligned} ds^2 &= P(r)dt^2-\frac{J(r,\theta)}{P(r)K(r)}dr^2-J(r,\theta)d\theta^2 \\ &-K(r)\sin^2\theta d\varphi^2 \end{aligned} \quad (10)$$

Unless for specific purposes, some metric coefficients: *P*(*r*), *J*(*r*,θ), and *K*(*r*) will be written as *P*, *J* and *K* respectively for the sake of brevity.

3.2. Dirac Particles Emission

In curved spacetime the Dirac equation is defined as

$$i\hbar\tilde{\gamma}^\mu D_\mu\psi=0 \quad (11)$$

where D_μ is the covariant derivative. It can be defined as $D_\mu=\partial_\mu+\frac{1}{4}\omega_\mu^{ab}\sigma_{ab}$. ω_μ^{ab} is the connection spin, which can be described as $\omega_\mu^{ab}=e_\nu^a\partial_\mu e^{\nu b}+e_\mu^a\Gamma_{\mu\sigma}^\nu e^{\nu b}=\frac{1}{2}\left(e^{\nu a}\partial_\mu e_\nu^b-e^{\nu b}\partial_\mu e_\nu^a\right)$. σ_{ab} corresponds to the commutator of Minkowskian spacetime gamma matrices, $\sigma_{ab}=\frac{i}{2}[\gamma_a,\gamma_b]$. To distinguish variables that related with Minkowskian spacetime and curved spacetime, two types of indices were used. The latin index indicates the local framework (Minkowksian) and the Greek index indicates the variable in curved spacetime. The relationship between the Minkowskian gamma matrices with the curved spacetime gamma matrices is $\tilde{\gamma}^\mu=e_a^\mu\gamma^a$. There are several expressions of the Minkowskian gamma matrices that satisfy $\{\gamma^a,\gamma^b\}=2\eta^{ab}I$. And, in this work, the equation can be written as

$$\begin{aligned} \gamma^0 &= \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix}; & \gamma^1 &= \begin{pmatrix} 0 & \sigma^3 \\ \sigma^3 & 0 \end{pmatrix}; \\ \gamma^2 &= \begin{pmatrix} 0 & \sigma^1 \\ \sigma^1 & 0 \end{pmatrix}; & \gamma^3 &= \begin{pmatrix} 0 & \sigma^2 \\ \sigma^2 & 0 \end{pmatrix}; \end{aligned} \quad (12)$$

where σ^k is the Pauli matrices. In this study, the non-zero components of tetrad were selected that satisfy $g_{\mu\nu}=\eta_{ab}e_\mu^a e_\nu^b$ as follows

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$$e_a^{\mu} = \mathrm{diag}\left(\frac{1}{\sqrt{P}},\ \sqrt{\frac{KP}{J}},\ \frac{1}{\sqrt{J}},\ \frac{1}{\sqrt{K}\sin\theta}\right)$$

(13)

Using the tetrad, the gamma matrices in the elliptical spacetime can be obtained as

$$\begin{aligned}\tilde{\gamma}^t &= \frac{1}{\sqrt{P}}\begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix};\ \tilde{\gamma}^r = \sqrt{\frac{KP}{J}}\begin{pmatrix} 0 & \sigma^3 \\ \sigma^3 & 0 \end{pmatrix} \\ \tilde{\gamma}^\theta &= \frac{1}{\sqrt{J}}\begin{pmatrix} 0 & \sigma^1 \\ \sigma^1 & 0 \end{pmatrix};\tilde{\gamma}^\varphi = \frac{1}{\sqrt{K}\sin\theta}\begin{pmatrix} 0 & \sigma^2 \\ \sigma^2 & 0 \end{pmatrix}\end{aligned} \quad (14)$$

Spinor wave function ψ has two spin states: spin-up and spin-down thus the ansatz of wave function for both states can be expressed as

$$\begin{aligned}\psi_\uparrow &= \begin{pmatrix} A \\ 0 \\ B \\ 0 \end{pmatrix}\exp\left[\frac{i}{\hbar}S[t,r,\theta,\varphi]\right] \\ \psi_\downarrow &= \begin{pmatrix} 0 \\ C \\ 0 \\ D \end{pmatrix}\exp\left[\frac{i}{\hbar}S[t,r,\theta,\varphi]\right]\end{aligned} \quad (15)$$

where $\psi_\uparrow$ corresponds to the spin-up and $\psi_\downarrow$ corresponds to the spin-down. *S* is its actions. However in this paper, just the spin-up case which was studied because the spin-down case is just analogous. The action in above definiton can be expanded in the order of Planck constant power. The expansion from this hypothesis can be written as

$$\begin{aligned}S[t,r,\theta,\varphi] &= S_0[t,r,\theta,\varphi] \\ &+ \sum_n \alpha_n \hbar^n S_n[t,r,\theta,\varphi]\end{aligned} \quad (16)$$

For *n* = 1, 2, 3, ..., α_n is a constants that keep all the expansion terms have the action dimension. Due to very small value of the Planck constant, the terms for $n \geq 1$ can be neglected. (Siahaan & Triyanta, 2010; Kerner & Mann, 2008). Substituting the action expansion into wave function in **equation (15)** and the Dirac in **equation (11)**, the solutions can be obtained. Based on this substitution, the expressed equations can be written as

$$\frac{iA}{\sqrt{P}}\partial_t S + B\sqrt{\frac{KP}{J}}\partial_r S = 0 \quad (17)$$

$$\frac{B}{J}\partial_\theta S + \frac{iB}{\sqrt{K}\sin\theta}\partial_\varphi S = 0 \quad (18)$$

$$-\frac{iB}{\sqrt{P}}\partial_t S + A\sqrt{\frac{KP}{J}}\partial_r S = 0 \quad (19)$$

$$\frac{A}{J}\partial_\theta S + \frac{iA}{\sqrt{K}\sin\theta}\partial_\varphi S = 0 \quad (20)$$

According to Hamiton-Jacobi method, the action can be expressed by the following components. Thus, the equation can be re-expressed as

$$S = -Et + R(r) + \Theta(\theta) + \Phi(\varphi) \quad (21)$$

where *E* is the energy, *R* is the radial component of action, Θ and Φ is component that correspond to azimuthal angle and polar angle respectively. If *A* and *B* are not zero and using the **equations (18)** and **(20)**, the solution leads to a single equation. Then, the equation can be written as

$$\partial_\theta \Theta = -\frac{i}{J\sqrt{K}\sin\theta}\partial_\varphi \Phi \quad (22)$$

The solution of above equation can be obtained through integration. Therefore, the integration process creates,

$$\Theta = \int_0^\pi -\frac{i\partial_\varphi \Phi}{J\sqrt{K}\sin\theta}d\theta = \frac{-2i}{J\sqrt{K}}\partial_\varphi \Phi \quad (23)$$

It is clear that Θ does not depend on either *A* or *B*. Furthermore, **equations (17)** and **(19)** have two possible solutions. The solution can be written as

1. For *A* = *iB*

$$\partial_r R = -\frac{E}{P}\sqrt{\frac{J}{K}} \quad (24)$$

2. For *A* = -*iB*

$$\partial_r R = \frac{E}{P}\sqrt{\frac{J}{K}} \quad (25)$$

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Thus, the correlation is

$$\partial_r R_{\pm} = \pm \frac{E}{P}\sqrt{\frac{J}{K}} \quad (26)$$

The plus sign in the above equation correspond to solution for outgoing particle and vice versa. By recalling the metric coefficient in **equations (7)-(9)**, the solution of *R* can be obtained according to integration. Thus, the equation can be written as

$$R_{\pm} = \int \frac{E}{P(r)}\sqrt{\frac{r^2 + a^2\cos^2\theta}{r^2 + a^2}}\,dr \quad (27)$$

For a static black hole, its particle emission occurs at the event horizon. Therefore, metric coeffiecient *P* using the Taylor expansion can be expanded. The equation can be written as

$$P(r) = P(r_h) + P'(r_h)(r - r_h) + O(r - r_h)^2 \quad (28)$$

The prime sign defines the derivative with respect to radial coordinate *r*. Thus, the integration of the above can be solved by

$$R_{\pm} = \pm\int_{r_h-\delta}^{r_h+\delta} \frac{E}{P'(r_h)(r - r_h)}\sqrt{\frac{r^2 + a^2\cos^2\theta}{r^2 + a^2}}\,dr$$

$$R_{\pm} = \left. \mp\frac{\sqrt{2}E\sqrt{\frac{i(r + ia\cos\theta)\csc^2\left(\frac{\theta}{2}\right)}{a + ir} \times \frac{(ir + a\cos\theta)\sec^2\left(\frac{\theta}{2}\right)}{a + ir}}}{P'(r_h)(a^2 + r_h^2)\sqrt{\frac{a^2 + 2r^2 + a^2\cos 2\theta}{a^2 + r^2}}\sqrt{\frac{(a - ir)\tan^2\frac{\theta}{2}}{a + ir}}} \times \left(\begin{array}{l} -2(a^2 + r_h^2)\,EllipticPi\left(-\cot^2\left(\frac{\theta}{2}\right), \sin^{-1}\left(\sqrt{\frac{(a + ir)\tan^2\frac{\theta}{2}}{a + ir}}\right), \cot^4\left(\frac{\theta}{2}\right)\right) + \\ (a^2 + 2r_h^2 + a^2\cos 2\theta)\,EllipticPi\left(\frac{(a + ir_h)\cot^2\frac{\theta}{2}}{a - ir_h}, \sin^{-1}\sqrt{\frac{(a - ir)\tan^2\frac{\theta}{2}}{a + ir}}, \cot^4\left(\frac{\theta}{2}\right)\right) + \\ a(a - ir_h)\,EllipticF\left(\sin^{-1}\left(\sqrt{\frac{(a - ir)\tan^2\frac{\theta}{2}}{a + ir}}\right), \cot^4\frac{\theta}{2}\right)\sin^2\theta \end{array} \right) \right|_{r_h-\delta}^{r_h+\delta} \quad (29)$$

where *Elliptical Pi* correspond to the third kind of elliptical integral (incomplete) and *Elliptical F* is the first kind of elliptical integral (incomplete). By applying the boundary condition for position around the event horizon, so it can be easily find out that the solution of *R* contain a variable that depends on θ, a and r_h. Thus, by comparing the above radial solution of action and the radial solution of action for spherical symmetric black hole,

$$R_{\pm} = \mp\frac{iE}{P'(r_h)}\xi(a, r_h, \theta) \quad (30)$$

where $\xi(a, r_h, \theta)$ is an function that summarize all variables that depend on a, r_h and θ. If the black hole is not too ellipse or it satisfy $a \ll r_h$, the term $\sqrt{\frac{r^2 + a^2\cos^2\theta}{r^2 + a^2}}$ can be

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expressed as $1+\frac{a^2\left(\cos^2\theta-1\right)}{2r^2}+O\left(a^4\right)$ and the integration in equation (29) can be solved as

$$R_\pm=\pm\int_{r_h-\delta}^{r_h+\delta}\frac{E}{P'\left(r_h\right)\left(r-r_h\right)}\left(1+\frac{a^2\left(\cos^2\theta-1\right)}{2r^2}\right)dr \\ =\mp\frac{i\pi E}{P'\left(r_h\right)}\left(1-\frac{a^2\sin^2\theta}{2r_h^2}\right) \tag{31}$$

From the above expression, it clear that the solution of *R* in the elliptical black hole depend on its azimuthal angle. For the spherically symmetric black hole, a = 0, the solution above will reduce to $R_\pm=\mp i\pi E/P'\left(r_h\right)$. This correlation is in a good agreement with previous calculations (Triyanta & Bowaire, 2013). In other words, for spherically symmetric case, the variable ξ is constant, (*i.e.* π).

In the general case, the solution for Θ and Φ in both outgoing particle and ingoing particle are constant. Thus, the expression of action can be written as

$$S_+=-Et+\frac{iE}{P'\left(r_h\right)}\xi\left(a,r_h,\theta\right)+\frac{-2i}{J\sqrt{K}}\partial_\varphi\Phi+\Phi \tag{32}$$

$$S_-=-Et-\frac{iE}{P'\left(r_h\right)}\xi\left(a,r_h,\theta\right)+\frac{-2i}{J\sqrt{K}}\partial_\varphi\Phi+\Phi \tag{33}$$

By recalling the spin-up wave function as **equation (15)** and if almost all ingoing particles are absorbed or $P_{in}=1$, the probability of outgoing particles leads to

$$P_{out}=\exp\left(\frac{-4E}{\hbar P'\left(r_h\right)}\xi\left(a,r_h,\theta\right)\right) \tag{34}$$

According to the Damour-Ruffini method, the spectrum of Hawking radiation for fermion emission as

$$N_B=\frac{P_{out}\Gamma}{1+P_{out}} \tag{35}$$

where Γ is the absorptivity of black hole. Since the probability of ingoing particles $P_{in}=1$, it clear that its absorptivity is equal to unity, Γ = 1. Therefore, by applying the probability of outgoing particles to **equation (35)** and by using the Boltzmann factor definition, the Hawking temperature for the elliptical black hole is

$$T=\frac{\hbar P'\left(r_h\right)}{4\xi\left(r_h,a,\theta\right)} \tag{38}$$

From the equation above, it clear that the temperature does not only depend on its horizon but also on the semi-principal axis a and the azimuthal angle θ. This result is quite interesting. Since its temperature depend on its azimuthal angle, every point on the black hole surface may have a different temperature. In other words, the elliptical black hole does not satisfy the equilibrium thermal conditions. This can be understood based on the relationship between temperature of a black hole and its surface gravity, $T=\kappa/2\pi$. Since every point in the surface has different distance to its center, the surface gravity of elliptical black hole also depends on θ. In addition, if the black hole satisfy spherically symmetric condition, $\xi=\pi$, its temperature will leads to $T=\hbar P'\left(r_h\right)/4\pi$.

4. CONCLUSIONS

By using the complexpath method, black hole temperature for an elliptical black hole due to the Dirac particles emission can be derived. From the result, it is clear that the temperature does not only depend on its horizon but also on the semi-principal axis and the azimuthal angle. The implication of this result is in an elliptical black hole its surface has heterogenous temperature or it does not satisfy equilibrium thermal condition.

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6. AUTHORS' NOTE

The author declare that there is no conflict of interest regarding the publication of this article. Author confirmed that the data and the paper are free of plagiarism.

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